6 Mean Molecular Weights

This problem serves to familiarize yourself with the important concept of mean molecular weights.

(1) Calculate and interpret the values of $\mu$ under the following circumstances:

(a) All hydrogen ($X = 1, Y = 0$), fully ionized
(b) All helium ($X = 0, Y = 1$), partial ionization (average charge state: He$^{1+}$)
(c) All helium ($X = 0, Y = 1$), fully ionized
(d) All metals ($X = 0, Y = 0$), fully ionized

(2) Calculate the mean molecular weight per electron $\mu_e$ for completely ionized conditions of

(a) all hydrogen
(b) all helium, also find cases where electron degenerate helium is important in stars
(c) $X = Z = 0.5$

(3) Assuming conditions under which the assumption

$$\mu \approx \frac{2}{1 + 3X + 0.5Y}$$

is valid, calculate the rate of change of the mean molecular weight

(a) with respect to the metal content $Z$, holding the hydrogen fraction constant \( \left( \frac{\partial \mu}{\partial Z} \right)_X \)
(b) with respect to the helium content $Y$, holding the metal fraction $Z$ constant \( \left( \frac{\partial \mu}{\partial Y} \right)_Z \)
(c) Use the results to interpret the change of $\mu$ and the pressure $P$ during hydrogen and helium burning.

7 Main sequence stars and polytropes

Assume a star with the properties $M = 30 M_\odot$, $R = 6.6 R_\odot$, $X = 0.7$ and $Y = 0.3$ has been observed.

(a) Calculate the central pressure $P_c$.

(b) Calculate the central temperature $T_c$.\(^*\)

Hint: You need to use the parameterized plot of $\beta$ from lecture 3 - slide 22.

8 Stellar energetics

In this exercise, we want to derive a stability criterium for stars.

(a) Show that the gravitational energy $\Omega = -G \int \frac{M_r}{r} dM_r$ can also be written as $\Omega = -3 \int P dV$. Hint: Use the hydrostatic equilibrium equation.

(b) We now introduce the mass-averaged temperature $< T > := \frac{1}{M} \int T dM_r$. Show that for an ideal gas:

$$M < T > = -\frac{1}{3} \frac{\Omega \mu}{N_A k}$$

\(1\)
(c) Now use the mass-specific heat capacity \( c_v = \frac{d}{dT} \left( \frac{\partial U}{\partial T} \right)_v \) to show that for a thin, spherical shell of the star, where you can assume \( c_v = \text{const.} \):

\[
U = c_v \int T dM_r
\]

(3)

(d) Use the results from (b) and (c) to show that the total energy of the star \( W = \Omega + U \) can be written as:

\[
W = -(3\gamma - 4)U
\]

where \( \gamma = c_p/c_v \).

(4)

(e) What would a value of \( \gamma < \frac{4}{3} \) mean for the stability of the star? Do you know stars/conditions where this can happen? Hint: Think about very massive stars.

9 Stellar Mass Squared

Use the hydrostatic equilibrium equation

\[
\frac{dP}{dr} = -G \frac{M_r}{r^2} \rho(r)
\]

(5)

and the definition of the infinitesimal mass of a spherical shell

\[
dM_r = 4\pi r^2 \rho(r) dr
\]

(6)

to show, that the mass-squared of a spherical fraction of the star enclosed within the radius \( r_i \) can be written as

\[
M^2(r_i) = \frac{8\pi}{G} \left[ 4 \int_0^{r_i} r P(r) r^3 dr - P(r_i) r_i^4 \right]
\]

(7)

Also give the relation for the total mass-squared of the star \( M^2(R) \).