5 Lane-Emden equation and stellar modelling

5.1 Polytrope with $n = 0$

Consider a stellar model with finite radius $R$ and constant density profile $\rho(r) = \rho_c$ (see exercise sheet 1). This corresponds to a polytrope of index $n = 0$.

(a) Ritter’s First Integral

Show, that in this case, the solution of the Lane-Emden equation

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\phi_n}{d\xi} \right) = -\phi_n^n$$

with the boundary conditions $\phi_n(0) = 1$ and $\phi'_n(0) = 0$ at the center, and $\phi_n(\xi_1) = 0$ at the surface is parabolic:

$$\phi_0(\xi) = 1 - \frac{\xi^2}{6}$$

(b) Another lower bound on the central pressure

In Problem 3, a lower bound on the central pressure was derived. A stronger lower limit can be obtained using the constant density model. Consider the Lagrangian form of the hydrostatic equilibrium equation

$$\frac{dP}{dM} = -\frac{GM_r}{4\pi r^4}$$

derived in Problem 2.3. Eliminate the variable $r$ and integrate the equation to obtain the pressure

$$P = P_c \left[ 1 - \left( \frac{M_r}{M} \right)^{2/3} \right]$$

Use this to give a lower limit on the central pressure $P_c$. Note that although this is a stronger lower limit than that of Problem 3, we did not actually prove that this is really a lower limit here.

5.2 Polytrope with $n = 1$: Ritter’s Second Integral

Consider a polytrope with index $n = 1$. Show, that in this case, the Lane-Emden Equation takes the form of the spherical Bessel differential equation

$$r^2 \frac{d^2 R(r)}{dr^2} + 2r \frac{R(r)}{r} + (k^2 r^2 - m(m + 1)) R(r) = 0$$

with the general solution

$$R(r) = A \cdot j_n(kr) + B \cdot n_m(kr)$$

where $j_n(kr)$ and $n_m(kr)$ are the spherical Bessel functions of first and second order respectively. Then, derive the solution $\phi_1(\xi)$ by applying the boundary condition $\phi_1(0) = 1$ to the general solution (6).

Hint: Find the coefficients $k$ and $m$ first, by comparing equations (1) and (5).
5.3 General calculations for Polytropes

(a) Density ratio

Show that the ratio of mean to central density for any polytropic index $n$ is given by

$$\frac{n}{\rho_c} = -3 \left( \frac{\phi'}{\xi} \right)_{\xi_*}$$  \hspace{1cm} (7)\]

where $\phi' := \frac{d\phi}{d\xi}$ and $\xi_* = \xi_1$ is the first zero of $\phi$ and thus describes the surface of the star.

(b) Central Pressure

Show, that the central pressure can be written as

$$P_c = \frac{1}{4\pi(n+1)(\phi_n')^2} \cdot \frac{GM^2}{R^4}$$  \hspace{1cm} (8)\]