THERMODYNAMIC PROPERTIES: EQUATIONS OF STATE
On the Road to Pressure Integral

Momentum transfer imparted to surface $dS$:

$$\Delta p = 2p \cos \theta \hat{z}$$
Volume of parallelepiped

\[ \text{Volume of parallelepiped} = \mathbf{v}_p dt \cdot \mathbf{n} dS \]

\[ = v_p \cos \theta \ dt \ dS \]

Particles within this volume, moving in direction of \( \mathbf{p} \), can strike the bottom surface \( dS \) in a unit time.
Pressure Integral

Number of all particles with momentum between $p$ and $p + dp$, $\theta$ and $\theta + d\theta$, $\phi$ and $\phi + d\phi$ striking $dS$ will be:

Number per unit vol with momentum $p$ \times Flux Volume \times Fraction of solid angle subtended by $dS$

Total hitting $dS = N_{hit} = n(p) \times v_p \cos \theta \, dt \, dS \times \frac{\Delta \Omega}{4\pi}$

$P = N_{hit} \frac{1}{dS} \frac{dp}{dt} = N_{hit} \frac{1}{dS} \frac{2p \cos \theta}{dt}$

$P = \int_0^{\infty} pv_p n(p) dp \int_0^{\pi/2} \cos^2 \theta \sin \theta \, d\theta$

$= \frac{1}{3} \int_0^{\infty} pv_p n(p) dp$
Electron in a box: Length L

Schroedinger Wave Equation: \[ -\frac{\hbar^2}{2m} \nabla^2 \Psi = \epsilon_n \Psi \]

\[ \epsilon_n = \frac{\hbar^2}{2m} \left( \frac{\pi}{L} \right)^2 \left( n_x^2 + n_y^2 + n_z^2 \right) \]
\[ = \frac{\hbar^2}{2m} \left( \frac{\pi}{L} \right)^2 n^2 \]

Fill the box with N electrons. Each \( \epsilon_n \) level can take 2 electrons (Pauli). Electrons fill box from lowest energy level up to a maximum energy level we call \( \epsilon_F \).

Only positive integers \( n_{x,y,z} \) are valid. States in the positive octant (1/8) of sphere in \( n_x, n_y, n_z \) coordinates contribute. Volume of this sphere: \( 4\pi n_F^3 / 3 \), \( n_F \) is the radius of sphere at the Fermi Level with energy \( \epsilon_F \). \( N \) can now be related to the number of levels

\[ N = 2 \times \frac{1}{8} \times \frac{4\pi}{3} n_F^3 = \frac{\pi}{3} n_F^3 \]
Energy of Electron Gas in Box

\[ U = 2 \sum_{n \leq n_F} \epsilon_n = 2 \times \frac{1}{8} \times 4\pi \int_0^{n_F} \epsilon_n n^2 \, dn \quad \epsilon_n = \frac{\hbar^2}{2m} \left(\frac{\pi}{L}\right)^2 n^2 \]

\[ = \frac{\pi^3}{10m} \left(\frac{\hbar}{L}\right)^2 n_F^5 = \frac{\pi^3}{10m} \left(\frac{\hbar}{L}\right)^2 \left(\frac{3N}{\pi}\right)^{5/3} \quad N = \frac{\pi}{3} n_F^3 \]

The pressure of this electron gas is then given by (and \( V = L^{1/3} \))

\[ P_e = -\frac{\partial U}{\partial V} = \frac{\pi^3}{15m} \hbar^2 \left(\frac{3n_e}{\pi}\right)^{5/3} \]

Where \( n_e = N/V \) is the electron number concentration.

Pressure **NOT** dependent on temperature, but only on the electron volume concentration! Degenerate electron gas does **NOT** behave like an ideal gas.
Relativistic Degenerate Electron Gas

We previously had on page 7 for kinetic energy of electron in orbital “n”:

\[ \epsilon_n = \frac{\hbar^2}{2m} \left( \frac{\pi}{L} \right)^2 n^2 \]

Let us equate this to \( p^2 / 2m \) and get:

\[ p = \hbar \pi n / L = \hbar \pi n / V^{1/3} \]

For the pressure integral, we need \( n(p) = \frac{dN}{V dp} \)

We also had, from page 7, for the total number of electrons in the box:

\[ N = \frac{\pi}{3} n_F^3 \]

These two equations give us \( p \) as a function of \( N \), by substituting for \( n \):

\[ p = \frac{\hbar \pi}{V^{1/3}} \left( \frac{3N}{\pi} \right)^{1/3} \]

Solve for \( N \), and take \( dN/dp \). Result is:

\[ \frac{dN}{V dp} = \frac{p^2}{\pi^2 \hbar^3} \equiv n(p) \]
For relativistic electrons, \( v \approx c \) and the pressure integral becomes:

\[
P_{e}^{rel} = \frac{1}{3} \int_{0}^{p_{F}} p c \frac{p^2}{\pi^2 \hbar^3} \, dp
\]

\[
= \frac{c}{12\pi^2 \hbar^3} p_{F}^4
\]

From previous page, we have all we need to get the Fermi momentum, \( p_{F} \)

\[
p_{F} = \hbar \pi^{2/3} (3n_{e})^{1/3}
\]

Finally, we have the relativistic degenerate electron pressure:

\[
P_{e}^{rel} = \frac{\hbar c \pi^{2/3}}{12} (3n_{e})^{4/3}
\]
THE PATH TO RADIATION PRESSURE
Photon Gas in Box: Length L

Maxell’s Equation for the Electric field:

\[ c^2 \nabla^2 E = \frac{\partial^2}{\partial t^2} E \]

\[ E_x = E_{x_0} \sin \omega t \ \cos \frac{n_x \pi x}{L} \ \sin \frac{n_y \pi y}{L} \ \sin \frac{n_z \pi z}{L} \]

With \( E_y \) and \( E_z \) given by similar expressions by cyclic shifting of the cosine in the right (\( \rightarrow \)) direction

Substitution of these back into Maxell’s Eqn yields:

\[ c^2 \pi^2 (n_x^2 + n_y^2 + n_z^2) = c^2 \pi^2 n^2 = \omega^2 L^2 \rightarrow \omega_n = \frac{n \pi c}{L} \]

Mode Energy:

\[ \epsilon_n = \hbar \omega_n = \frac{n \hbar \pi c}{L} = \frac{n \hbar \pi c}{\sqrt[3]{1}} \]
Each mode will have some average number of photons in it. Call this number $n$. We need to eventually find a formula for $S$.

Total energy of the photon gas: 
$$U = \sum_{n=1}^{\infty} s_n \epsilon_n = \sum_{n=1}^{\infty} \hbar s_n \omega_n$$

Photon pressure:
$$P_\gamma = \frac{\partial U}{\partial V} = \sum_{n=1}^{\infty} \hbar s_n \frac{\partial \omega_n}{\partial V} = \frac{1}{3V} \sum_{n=1}^{\infty} \hbar s_n \omega_n$$
$$= \frac{U}{3V}$$

Thermodynamic Partition Function $Z$ for a mode (review your thermodynamics):
$$Z = \sum_{s=0}^{\infty} \exp(-s\hbar \omega/\tau) = \sum_{s=0}^{\infty} x^s, \ x \equiv \exp(-\hbar \omega/\tau)$$
$$= \frac{1}{1-x} = \frac{1}{1-\exp(-\hbar \omega/\tau)}$$

Energy in a mode:
$$\langle \epsilon_n \rangle = \tau^2 \frac{\partial}{\partial \tau} \ln Z$$
$$= \frac{\hbar \omega}{\exp(\hbar \omega/\tau)-1}$$
Total Energy: \[ U = \sum_n \langle \epsilon_n \rangle = \sum_n \frac{\hbar \omega_n}{\exp(\hbar \omega_n / \tau) - 1} \]

Remember from page 12, that \( n \) is a triplet of integers: \( n_x, n_y, n_z \). We replace the sum by an integral over \( dn_x, dn_y, dn_z \), and change to spherical coordinates:

\[ U = 2 \times \frac{1}{8} \int_0^\infty 4\pi n^2 dn \hbar \frac{\omega_n}{\exp(\hbar \omega_n / \tau) - 1} \]

2 polarizations

And we recall from page 12 that \( \omega_n = n\pi c / V^{1/3} \)

\[ U = \pi^2 \hbar c / V^{1/3} \int_0^\infty n^3 dn \frac{1}{\exp(\hbar \omega_n / \tau) - 1} \]

\[ = (\pi^2 \hbar c / L)(\tau L / \pi \hbar c)^4 \int_0^\infty dx \frac{x^3}{\exp x - 1} \]

\[ = \frac{\pi^2 V}{15 \hbar^3 c^3} \tau^4 \]

Change variable \( x = \pi \hbar c n / L\tau \)

\[ \pi^4 / 15 \]
Pressure (at last!) was found, on page 13, to be $U/3V$:

$$P_\gamma = \frac{\pi^2}{45 \hbar^3 c^3} \tau^4$$
Hydrostatic Equilibrium

The gravitational body forces integrated over the volume \( V \) must be balanced by the pressure acting on the total surface area \( A \).

\[
\int_V g \rho \, dV = \int_A P \, da \quad (*)
\]

Gauss’ Law (Divergence Theorem): \( \int_V (\nabla \cdot F) \, dV = \int_A F \cdot da \)

Use \( F = f \mathbf{v} \), where \( \mathbf{v} \) is a constant vector (has no \((x,y,z)\) dependence) and also use the following identity: \( \nabla \cdot (f \mathbf{v}) = f (\nabla \cdot \mathbf{v}) + \mathbf{v} \cdot \nabla f \)

Now Div. Thm: \( \int_V (\nabla \cdot f \mathbf{v}) \, dV = \mathbf{v} \cdot \int_V \nabla f \, dV = \mathbf{v} \cdot \int_A f \, da \)

\[ \Rightarrow \mathbf{v} \cdot \left( \int_V \nabla f \, dV - \int_A f \, da \right) = 0 \]

But \( \mathbf{v} \) was chosen to be a constant (non-zero) field. So, term in ( ) is zero. Use it on RHS in force balance equation (*).
Force balance: \[ \int_V \mathbf{g} \rho \, dV = \int_A P \, dA \]

\[ = \int_V \nabla P \, dV \quad \text{(From last identity on previous page)} \]

\[ \Rightarrow \int_V \mathbf{g} \rho \, dV - \int_V \nabla P \, dV = 0 \]

This result must hold throughout the entire volume \( V \), and since volume shape is arbitrary, we therefore have:

\[ \Rightarrow \mathbf{g} \rho = \nabla P \]

In spherical geometry, using \( g = -G \frac{M}{r^2} \), where \( M \) is the enclosed mass inside radius \( r \), we finally have:

\[ \frac{dP}{dr} = -G \frac{M(r)}{r^2} \rho(r) \]
Summary of Results so far.

\[ \frac{dP}{dr} = -G \frac{M(r)}{r^2} \rho(r) \]

\[ P_{\text{gas}} = n \tau, \quad n = \frac{N}{V} \]

\[ P_e = -\frac{\partial U}{\partial V} = \frac{\pi^3}{15m} \bar{h}^2 \left( \frac{3n_e}{\pi} \right)^{5/3} \]

\[ n_e = \frac{N_e}{V} \]

\[ P_{e\text{rel}} = \frac{\bar{h} c \pi^{2/3}}{12} \left( 3n_e \right)^{4/3} \]

\[ P_\gamma = \frac{\pi^2}{45\bar{h}^3 c^3 \tau^4} \]